PROMYS 2014 Research Project on Portfolio Theory – Jonathan Hanke

1. Introduction to Portfolio Theory

Portfolio theory is a study of how to allocate assets in a given market environment to best achieve investment goals. (Here a **market** is defined to consist of a given collection of assets one can purchase at some given price, whose prices will change over time.) Mathematically speaking, one usually begins with certain assumptions (i.e. definitions) of what the market conditions will be in terms of "probabilities", and then one defines an **investment strategy** to be a rule for allocating assets in that market when one is given the opportunity to do so.

The simplest example of an investment strategy (and also the most common) is the **buy-and-hold strategy** where the portfolio manager buys a fixed amount of each asset at time $t = 0$, and then waits to see what happens at some later time. This is often referred to as a passive strategy. An example of a more active investment strategy is the **greedy momentum strategy** which puts all of our value into the asset whose price increased by the largest factor since the last time we looked. The greedy momentum strategy might be used by someone who believes that things that increase in value will continue to increase in value, and wanted to maximize their total portfolio value over time. Another (active) investment strategy is the **constant weighted strategy** where fixed percentages of our total portfolio are allocated to each asset in the market. For example in a market with assets reflecting stock in Coke and Pepsi, a Coke enthusiast might allocate 75% to Coke and 25% to Pepsi, though a Pepsi enthusiast might allocate 25% to Coke and 75% to Pepsi. The simplest constant weighted portfolio is the **equal weighted portfolio**, where each asset is given the same percentage of the portfolio assets whenever possible.

There are many types of investment strategies one can consider, and their "success" depends on what you, the portfolio manager, are trying to accomplish. Sometimes you want to achieve the largest possible portfolio value, sometimes you want to lose the least, and sometimes you just don't want things to change too much (so no one gets nervous)! Remember that *nothing is certain in the real world* and pretty much anything can happen, so the best we can do is to understand the *probabilities* that what we want to occur actually happens. This may be a bit unsettling at first, but there is a lot of really cool mathematics that happens in the language of probability (which is a mathematical discipline of its own!), and there is a lot one can say for certain (i.e. give exact probabilities and precise long-term behavior) in this language.

In this reseach project, your goal is to try to analyze the behavior of various portfolio strategies in some simple markets (that are actually not too far from the real world), and decide how well these accomplish the goals for the portfolio.

2. Probability

Probability is the mathematics of chance. It is based on a given event having a specific outcomes each of which (we postulate) occurs a certain percentage of the time. Another perspective on this is that if we could repeat the event many many times, then the ratio of the number of times the outcome occurs to the number of times the event was performed approaches a well-defined number which we call the **probability** of that outcome. The most common examples of an event are: the flip of a **fair coin**, which we define to have a 50% chance of ending up on either side, and (more generally) the **fair n-sided die** which when rolled has a $\frac{1}{n}$ probability of giving any of the n sides as an outcome.

The list of outcomes of any event (which we always take to be a set of real numbers), together with their associated probabilities, is defined to be a **random variable**. The only requirements are that the probabilitles are all non-negative real numbers, and the sum of all probabilities (for all outcomes of the random variable) is one. Usually if X is a random variable, then we can talk about the outcomes i of X, and the probability $p(i)$ for each outcome i to occur. In this language, we require that $0 \leq p(i) \leq 1$ for each i and also that the sum $\sum_i p(i) = 1$.

For a random variable X, it is useful to talk about the **expected value** $\mathbb{E}[X]$ of X, defined by

$$\mathbb{E}[X] := \sum_i i \cdot p(i).$$

Intuitively, the expected value $\mathbb{E}[X]$ tells us the average value of the random variable if we performed the event many many times.

(**Note:** It is possible that there are not just finitely many outcomes for a random variable, in which case all sums should be replaced with integrals! But don't worry – it is not necessary to deal with these for this project. We'll usually stick to random variables that come in one way or another from flipping a coin.)

3. Markets and Portfolios

In order to talk about portfolios, we need to have a market of assets to choose from. Mathematically, we will describe this by giving an initial set of (positive) prices at time t_0 and a random variable X_a to describe the price P_a of each asset a at a given later time t_1. Since we choose how the portfolio is distributed in each of the assets at the initial starting time, the inital prices are not relevant and we take $P_a := 1$. Also, since prices are always assumed to be positive in a market where stocks don't disappear, it is often convenient to replace the price random variables X_a by the price multiplier random variables R_a defined by $R_a := \frac{X_a}{P_a}$.

There are many ways to model a market, but we will consider multiplicative models based on a coin flip. For simplicity we will assume that there are only two assets (e.g. Coke and Pepsi) and that time moves in discrete units (say seconds). For each unit of time, we will flip a fair coin. If heads comes up, then we double the price of the first asset and half the price of the second asset. If tails comes up then we do the opposite (i.e. halving the first price and doubling the second price). We call this an **antisymmetric two stock market** and will be interested in the behavior of each of our portfolio strategies in this market. We could contrast this to the **symmetric two stock market** where if heads comes up then both prices double, and if tails comes up then both prices go down. Since we have two stocks, it is convenient to refer to the prices collectively by a **price pair** (X_1, X_2) where X_i is defined to be the price of the i-th asset.

For example, if at time $t = 0$ the stocks have the same starting price of 1 (which we can always assume without loss of generality) and the coin flip gives heads (denoted by "H"), then the price pair at time $t = 1$ for the antisymmetric market will be $(2, \frac{1}{2})$ and $(2, 2)$ for the symmetric market. For a sequence of coin flips given by HHTH the anti-symmetric market will have price pairs given by

Time	$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$
Symmetric market prices	$(1, 1)$	$(2, 2)$	$(4, 4)$	$(2, 2)$	$(4, 4)$
Anitsymmetric market prices	$(1, 1)$	$(2, \frac{1}{2})$	$(4, \frac{1}{4})$	$(2, \frac{1}{2})$	$(4, \frac{1}{4})$

4. Questions

4.1. **Warmup Questions.** Try to understand the answers to each of the following questions in the symmetric market and the antisymmetric market.

(1) **(Equal buy-and-hold portfolio)** What is the expected value after t seconds have passed of a portfolio that starts with \$1 and distributes it equally among both assets at $t = 0$?

(2) **(General buy-and-hold portfolios)** What is the expected value after t seconds of general constant weighted portfolio that at $t = 0$ puts $x\%$ of the portfolio in the first asset and $(1 - x)\%$ in the second asset?

(3) **(Equal-weighed portfolio)** What is the expected value after t seconds of the equal-weighted portfolio?

(4) **(Constant-weighed portfolios)** What is the expected value after t seconds of general constant weighted portfolio that at every time $t \in \mathbb{Z} \geq 0$ puts $x\%$ of the portfolio in the first asset and $(1 - x)\%$ in the second asset?

4.2. **Open-ended Questions.**

(1) **(Asymptotic Behavior as $t \to \infty$)** Can we make a statement about the expected value of each of the warmup question portfolios at time t, where t is very large?

(2) **(Markets based on unfair coins)** What happens to the behavior of the warmup question portfolios at time t if we instead use markets based on coins which have a probability p_H of coming up heads, and $p_T = 1 - p_H$ of coming up tails?

(3) **(Markets with two independent coins)** What happens to the behavior of the warmup question portfolios at time t if we instead use markets where the price of each asset will double or half at the next time based on its own fair coin?

(4) **(Markets with three independent coins)** What happens to the behavior of the warmup question portfolios after time t if we instead use markets with three assets where the price of each asset will double or half at the next time based on its own fair coin?

4.3. **Advanced Questions and Theorems.**

(1) **Addition as convolution** Suppose you have two independent random variables X and Y with respective probability distributions p_X and p_Y. Show that the random variable $X+Y$ (defined by computing the probabilities of the possible values of the sum of the two values of X and Y weighted by their probabilities) has probability distribution for each outcome c given by

$$p_{X+Y}(z) = \sum_{x+y=z} p_X(x) \cdot p_Y(y) = \sum_x p_X(x) \cdot p_Y(z-x).$$

(Here the sum is replaced by an integral if we use a probability density instead of discrete outcome probilities.)

(2) **The Law of Large Numbers** Suppose you have n independent random variables $X_1, X_2, \ldots, X_n$ with the same probability distributions $p_X(x)$ as some random variable X. Then as $n \to \infty$ we have that

$$\frac{X_1 + X_2 + \cdots + X_n}{n} \to \mathbb{E}[X],$$

where here $E[X]$ is the constant random variable that has value $E[X]$ with probability one.

(3) **The Central Limit Theorem** Suppose you have n independent random variables $X_1, X_2, \ldots, X_n$ with the same probability distributions $p_X(x)$ as some random variable X, and also $\mathbb{E}[X] = 0$. Then as $n \to \infty$ we have that

$$\frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}} \to G_{\mathrm{Var}(X)},$$

where here G_{σ^2} is the **Gaussian random variable** with variance σ^2 which is defined by the probability density

$$P_{G_{\sigma^2}}(x) := \tfrac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-x^2}{2\sigma^2}},$$

and the **variance** $\mathrm{Var}[Y]$ of a random variable Y is defined by

$$\mathrm{Var}(Y) := \sum_y p_Y(y)^2.$$

(Note that the variance of G_{σ^2} is σ^2 and that the variance is always ≥ 0. This helps to explain why we write the variance as σ^2.)

(4) **(Markets with continuous time)** It is customary to make a continuous time model for markets based on coin flips by imagining that as the time interval gets subdivided into r pieces, and the multiplier for that interval gets multiplied by $\sqrt{r}$. This leads to the notion of a **Gaussian random variable** $G_{\Delta t}$ with **variance** Δt, over a time interval from t to $t + \Delta t$ we can imagine the market evolving by multiplying (or dividing) the appropriate prices by the random variable $e^{G_{\Delta t}}$. How do the portfolios above behave in the associated continuous time markets?

5. Some Further References

Below is a selection of references for further reading on the topics of probability and its application to various market models and portfolio theory that I have found useful. The first section gives purely mathematical references, while the second section has references that gives some application of models similar to those developed in this project to asset pricing and portfolio theory.

(1) **An Introduction to Measure Theory** by Terrence Tao – A very accessible first book on measure theory that exposes some of the subtleties of the various notions of measure.

(2) `http://terrytao.wordpress.com` blog by Terrence Tao – There are many wonderful topics discussed here. For probability, there is a nice "Review of Probability theory" post from January 2010.

(3) **Probability and Measure** by Patrick Billingsley – An classic introduction to the fundamental concepts and theorems about probability and its connection with measure theory.

(4) **A Course in Probability Theory (3rd ed)** by Kai Lai Chung – Another classic introduction to the fundamental concepts and theorems about probability and its connection with measure theory.

(5) **Brownian Motion and Stochastic Calculus (2nd ed)** by Ioannis Karatzas and Steven E. Shreve – This is a very advanced graduate reference that gives a precise description of random variables and stochastic calculus. (*Reading Requirements:* A strong foundation in measure theory and the measure theoretic-formulation of random variables and stochastic processes.)

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(6) **Stochastic Calculus for Finance I and II** by Steven E. Shreve – These books give an introduction to the formalism of Brownian motion, Itô calculus and Stochastic differentials (among other things) from a discrete time coin-flipping (binomial) price model. (*Reading Requirements:* Familiarity with basic concepts in probability.)

(7) **Stochastic Portfolio Theory** by E. Robert Fernholz – This advanced book gives a stochastic differential description of the value (process) of various instantaneously rebalanced portfolio strategies in a markets described by a large class of contonuous time markets whose prices are related to geometric Brownian motion by reasonable functions. (*Reading Requirements:* stochastic differentials, stochastic processes, and Itô calculus.)