Exam #1 for Math 2260 September 16, 2009

NAME (*please print*): Answer Key

HONOR CODE PLEDGE:

SIGNATURE: Jonathan Hanke

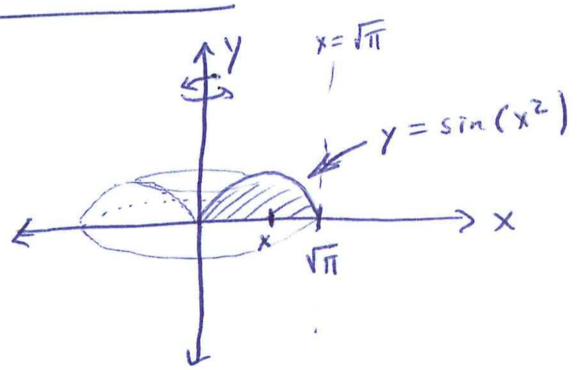
Please write your answers clearly to all problems, showing all work. You are not allowed to use any notes or review sheets or calculators during the exam. You have exactly 50 minutes to complete the exam. Good Luck!

Problem Number	Possible Points	Points Earned:
1	20	
2	20	
3	20	
4	20	
5	20	
6	(10)	
Total:	100	

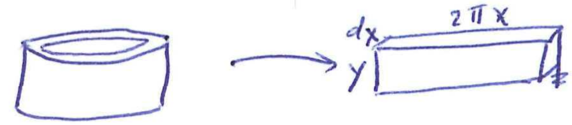
 Name: ____________________

(20) 1. Please find the volume of revolution obtained by rotating the region in the first quadrant bounded by $x = \sqrt{\pi}$ and the graph $y = \sin(x^2)$ about the y-axis.

Picture:



Slice / Shell at x:



$\Rightarrow$ Volume of shell at y $= 2\pi x y\, dx$

$= 2\pi x \sin(x^2)\, dx$

$$\Rightarrow \text{Total Volume} = \int_{x=0}^{\sqrt{\pi}} 2\pi x \sin(x^2)\, dx$$

$$= \pi \int_{x=0}^{\sqrt{\pi}} 2x \cdot \sin(\underbrace{x^2}_{u})\, dx$$

$$\left(\begin{array}{l} \text{Let } u = x^2 \\ \Rightarrow du = 2x\, dx \end{array} \right)$$

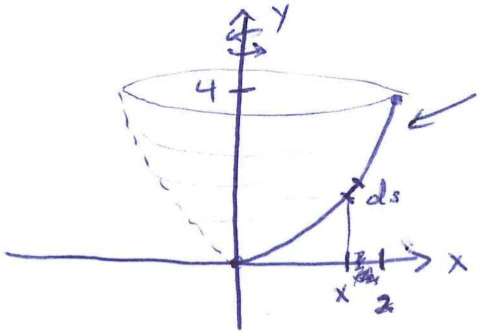
$$= \pi \left(-\cos(x^2) \right) \Big|_{x=0}^{\sqrt{\pi}}$$

$$= \pi \left(-\cos(\pi) + \cos(0) \right)$$

$$= \pi(- -1 + 1) = \boxed{2\pi}$$

 Name: ______________________

(20) 2. Please find the area of the surface of revolution obtained by rotating the portion of the graph $y = x^2$ with $0 \le x \le 2$ about the y-axis.



$y = x^2 = f(x) \Rightarrow f'(x) = 2x$

$$\Rightarrow \text{Surface Area} = \int_{x=0}^{2} 2\pi R \, ds$$

$$= \int_{x=0}^{2} 2\pi x \sqrt{1+(2x)^2} \, dx$$

$$= 2\pi \int_{x=0}^{2} x \sqrt{1+4x^2} \, dx$$

$$= \frac{2\pi}{8} \int_{x=0}^{2} 8x \sqrt{1+4x^2} \, dx$$

$$= \frac{\pi}{4} \left(\frac{2}{3} (1+4x^2)^{3/2} \right) \Big|_{x=0}^{2}$$

$$= \frac{\pi}{4} \cdot \frac{2}{3} \left((17)^{3/2} - (1)^{3/2} \right)$$

$$= \boxed{\frac{\pi}{6} \left((17)^{3/2} - 1 \right)}$$

Name: ______________________________

(20) 3. The radioactive element Calculonium decays to 10% of its starting amount after 2 days. What is the half-life of Calculonium, and how much will we have after 5 days if we initially start with 3 kilograms?

$A(t)$ = amount of Calculonium after t days

$$A(t) = A_0 e^{kt}$$

10% after 2 days $\Rightarrow$ $A(2) = A_0 e^{k \cdot 2} = \frac{1}{10} A_0$

$$\xRightarrow{\cdot \frac{1}{A_0}} e^{k \cdot 2} = \frac{1}{10}$$

$$\xRightarrow{\ln(\cdot)} 2k = \ln\left(\frac{1}{10}\right) = -\ln(10)$$

$$\Rightarrow k = \frac{-\ln(10)}{2}.$$

Start with $A_0 = 3$ lbs $\Rightarrow$ $A(t) = 3 \cdot e^{-\frac{\ln(10)}{2} t}$

Amount after 5 days $= A(5) = 3 \cdot e^{\frac{-\ln(10)}{2} \cdot 5}$

$$= 3\left(e^{\ln\left(\frac{1}{10}\right)}\right)^{5/2}$$

$$= \boxed{3 \cdot 10^{-5/2} \text{ lbs.}}$$

(20) 4. Please use your knowledge of calculus to derive the general solution $y(t)$ of the differential equation

$$y\frac{dy}{dt} = k(y^2+1)$$

where k is some arbitrary constant. What do all solutions $y(t)$ look like which satisfy $y(0) = 2$?

Seperate Variables $\Rightarrow \quad \frac{y}{y^2+1}\,dy = k\,dt$

Indefinite Integrals $\Rightarrow \quad \int \frac{y}{y^2+1}\,dy = \int k\,dt$

$$\frac{1}{2}\ln|y^2+1| = kt + c$$

$(y^2+1>0$ always$)$ $\Rightarrow \frac{1}{2}\ln(y^2+1) = kt + c$

$$\Rightarrow \ln(y^2+1) = 2kt + c'$$

$$\Rightarrow y^2+1 = e^{2kt+c'} = Ae^{2kt}$$

$$\Rightarrow y = \pm\sqrt{Ae^{2kt} - 1}$$

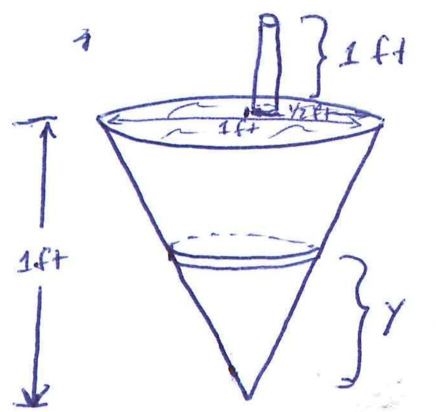
When $t=0$ we have $y(0) = +2 = \pm\sqrt{Ae^{2k\cdot 0} - 1}$

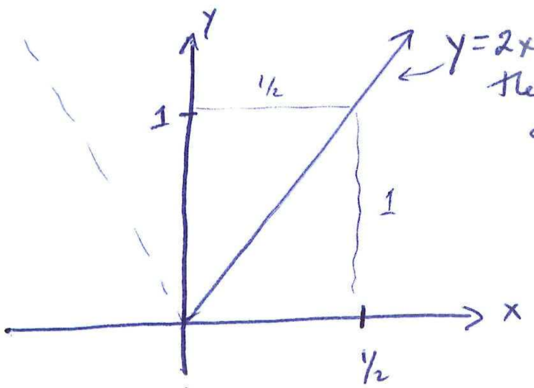
$2>0 \Rightarrow +$ sign and also $A \cdot 1 - 1 = 4$

$\Rightarrow A = 5$

$$\Rightarrow \boxed{y(t) = \sqrt{5e^{2kt} - 1}}$$

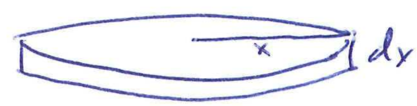
(20) 5. A milkshake fills a conical cup (with point down) of height 1ft and top diameter 1ft. Assuming that the density of the milkshake is 10 lbs/ft^3 and there is no friction, how much work does it take to drink the milkshake from a straw extending 1 ft above the top of the cup?





$y = 2x$ is the equation of the cone on the right in the x-y plane

Slice at height y:



$\Rightarrow$ Volume of slice at y $= \pi x^2 dy = \pi \left(\frac{y}{2}\right)^2 dy$
$= \frac{\pi}{4} y^2 dy \ (ft^3)$

$\Rightarrow$ Weight of slice at y $= 10 \cdot \frac{\pi}{4} y^2 dy \ (lbs)$

$\Rightarrow$ Work done to drink the slice at y $= (2-y) \cdot \frac{10\pi}{4} y^2 dy \ (ft \cdot lbs)$

$\Rightarrow$ Total Work $= \int_{y=0}^{1} (2-y) \frac{10\pi}{4} y^2 dy \ (ft \cdot lbs)$

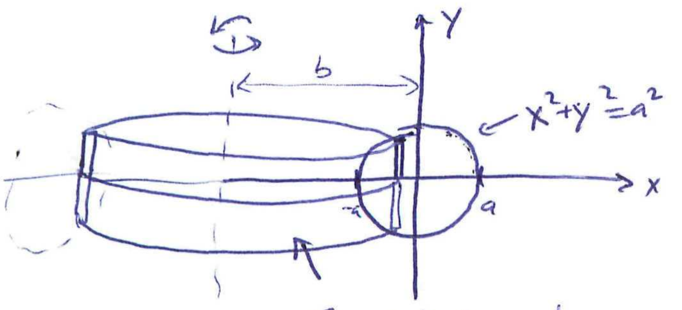
$$= \frac{5\pi}{2} \int_{y=0}^{1} (2y^2 - y^3)\,dy$$

$$= \frac{5\pi}{2} \left(\frac{2}{3}y^3 - \frac{y^4}{4} \right) \Big|_{y=0}^{1}$$

$$= \frac{5\pi}{2} \left(\left(\frac{2}{3} - \frac{1}{4} \right) - (0-0) \right)$$

$$= \frac{5\pi}{2} \cdot \left(\frac{8}{12} - \frac{3}{12} \right) = \boxed{\frac{25\pi}{24} \text{ ft-lbs}}$$

(10) 6. Please find formulas for the volume and surface area of a donut given by revolving a circle of radius a about an axis a distance b away from its center.



above x-axis $\Rightarrow y=\sqrt{a^2-x^2}$ gives the graph.

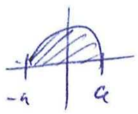
This shell has volume

$$2\pi r\cdot y = 2\pi(b+x)\cdot 2\sqrt{a^2-x^2}\,dx$$

Volume by Shells:

$$= \int_{x=-a}^{a} 4\pi\,(b+x)\sqrt{a^2-x^2}\,dx$$

$$= 4\pi\int_{x=-a}^{a}(b+x)\sqrt{a^2-x^2}\,dx$$



$$= 4\pi b\int_{x=-a}^{a}\sqrt{a^2-x^2}\,dx + 4\pi\int_{x=-a}^{a}x\sqrt{a^2-x^2}\,dx$$

$$= 4\pi b\cdot\frac{\pi a^2}{2} + 4\pi\cdot\frac{-1}{2}\cdot\int_{x=-a}^{a}-2x\sqrt{a^2-x^2}\,dx$$

This is a symmetric integral of an odd function! ☺

$$= 2\pi^2a^2b - 2\pi\left(\tfrac{2}{3}(a^2-x^2)^{3/2}\right)\Big|_{x=-a}^{a}$$

$$= 2\pi^2a^2b - \frac{4}{3}\pi\left((0)^{3/2}-(0)^{3/2}\right)$$

$$= \boxed{2\pi^2a^2b}$$