# Final Project: Analysis and Sums of 4 squares (Siegel's theorem)

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**Abstract**

It is a classical fact that every positive integer is a sum of 4 squares, and there are a variety of proofs explaining why. Above, we describe how one can use modular arithmetic and a certain kind of local-global principle to show this. However, it is possible to use a similar idea to also answer the deeper question of *how many ways* can we write any number as a sum of 4 squares? To do this, we need to keep track of the number of ways this can be done mod $p$ for all primes $p$, and do a little extra work at the prime 2. These local computations mod $p$ are computed using a very interesting sum called a *Gauss sum* and yield very simple formulas for the number of solutions mod $p$. Then, by multiplying these factors from each prime together, we can find an exact formula for the number of ways of writing any positive integer as a sum of 4 squares. Your project should carry this through, and perhaps find an exact formula for the number of representations of another interesting quadratic form. **[4-5]**

The following is a rough outline which may be useful in thinking about/organizing your project. Good general references are [1, §3.4, §9.1, §9.5–6, Appendix B3], and parts of [2]. If you have any questions about your project and/or readings, feel free to let me know, and we can setup a time to talk about it. Have Fun! =)

1. **Class numbers and reduction theory**
2. **Gauss sums and solutions over** $\mathbb{Z}_p$
3. **Hensel's lemma and solutions over** $\mathbb{Z}_{p^n}$
4. **Computing local densities at primes**
5. **Computing the local density at** $\infty$
6. **Zeta functions and the infinite product**

## References

[1] J. W. S. Cassels. *Rational quadratic forms*, volume 13 of *London Mathematical Society Monographs*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1978.

[2] John Milnor and Dale Husemoller. *Symmetric bilinear forms*. Springer-Verlag, New York, 1973. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 73.