Research Description

Jonathan Hanke

My research focuses on understanding the relationship between classical counting problems and their automorphic generating "theta" functions, specifically focusing on quadratic forms and theta series. This connection provides a great deal of information in both directions. For example, it can be used to determine the numbers represented by a given quadratic form and to prove explicit finiteness theorems for a quadratic form to represent a given set of numbers, and also to understand the structure of theta series as modular forms. An important tool in such investigations are explicit mass formulas, which provide a quantitative perspective for understanding "local-global" questions that can be applied to study many related arithmetic questions. My research program focuses on developing these tools further and applying them to understanding interesting counting problems, both theoretically and computationally.

1 Recent Work

1.1 Enumeration of Maximal Quadratic forms in $n \geq 3$ variables of bounded class number

The paper [Ha13] gives explicit bounds for enumerating all (classes of primitive) maximal positive definite quadratic forms Q_{Max} of bounded class number over $\mathbb{Z}$ in $n \geq 3$ variables, and provides a complete list of the 115 such forms of class number one. To perform this enumeration, the proposer has designed and implemented an extensive open-source software package [Ha14] for computing with Quadratic Spaces and Quadratic Lattices in Python, which is built upon the author's Quadratic Forms software [Ha6] distributed in Sage [Sage].

The bounds for these class numbers are derived from the exact formula [GHY] for the **mass**

$$\mathrm{Mass}(Q) := \sum_{\mathrm{Cls}(Q') \in \mathrm{Gen}(Q)} \frac{1}{\#\mathrm{Aut}(Q')}$$

of (the genus of) maximal definite quadratic forms $\mathrm{Gen}(Q_{\mathrm{Max}})$, which is given as a product of special values of the Riemann zeta function and possibly the special value of a Dirichlet L-function $L(s, \chi)$ with quadratic character χ (present when n is even), with rational adjustment factors $\lambda_p(Q_{\mathrm{Max}})$ at finitely many primes $p \in \mathbb{N}$ depending only on the local quadratic space $(\mathbb{Q}_p^n, Q_{\mathrm{Max}})$. The mass provides a lower bound for the class number $h(Q)$ of positive definite forms Q through the inequality $\mathrm{Mass}(Q) \leq \frac{h(Q)}{2}$, and our bound is obtained by providing a lower bound for the special values and all adjustment factors $\lambda_p(Q_{\mathrm{Max}})$ in terms of the determinant $\det(Q_{\mathrm{Max}})$.

To enumerate all maximal forms of bounded class number form these bounds, we compute all local quadratic spaces (V_p, Q_p) that could be associated to a small class number maximal quadratic form, and then explicitly construct all global rational quadratic spaces (V, Q) made from these (V_p, Q_p) locally at all primes $p \mid 2\det(Q_{\mathrm{Max}})$. Finally, for each such quadratic space a maximal quadratic lattice must be found and its class number must be computed. The algorithms for finding a maximal quadratic lattice L involve many explicit computations with the quadratic discriminant module $L^{\#}/L$, and are described in the paper [Ha16].

This work establishes a freely available open-source infrastructure suitable for further exploration of quadratic forms and quadratic lattices with a priori bounded class number, which is discussed further in Section 2.2.

1.2 Explicit Representability of Numbers

Paper [Ha2] uses the theory of modular forms to provide practical conditions for finding the numbers m represented by a positive definite integer-valued quadratic form Q in $n \geq 3$ variables. This allows one to compute the numbers represented by Q when $n \geq 4$, and also when $n = 3$ provided we restrict ourselves to numbers m within a fixed square-class $t\mathbb{Z}^2$ which is not of exceptional-type.[1]

This is proved by analyzing the theta function $\Theta_Q(z)$ as a sum of an Eisenstein series $E(z)$ and a cusp form $f(z)$, and computing explicit lower and upper bounds for the growth of the Eisenstein and cuspidal coefficients respectively. While the final statments are for quadratic forms over $\mathbb{Q}$, the necessary local computations (at all primes) here are done over a totally real number field in preparation for future work. The local factors are understood and computed using an explicit reduction procedure, which lends itself to quick computations. We use these results to prove the long-standing conjecture (due to Kneser and popularized by Kaplansky) that $x^2 + 3y^2 + 5z^2 + 7w^2$ represents all positive integers except 2 and 22.

1.3 The 290-Theorem and Universal Quadratic Forms

For convenience, we say that a positive definite quadratic form Q is **universal** if it represents all positive integers. The characterization of universal forms has a long history going back to Lagrange [L] and Ramanujan [Ra], though until recently it was incomplete. A notable recent development along these lines is the Conway-Schneeberger 15-Theorem [Con, Schn, Bh] which states that any positive definite form with even mixed terms (i.e. classically integral) is universal if and only if it represents the numbers 1, 2, 3, 5, 6, 7, 10, 14, and 15. The proof of this elegant theorem enables one to prove that there are exactly 204 (positive definite) classically integral universal quaternary forms. In 1993, Conway also formulated a more general "290-Conjecture" for all integer-valued positive definite forms. In [Bh-Ha] (joint with M. Bhargava) we prove this more general 290-Theorem, which states that an integer-valued positive definite quadratic form is universal if and only if it represents the 29 critical numbers

$$1, 2, 3, 5, 6, 7, 10, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29,$$
$$30, 31, 34, 35, 37, 42, 58, 93, 110, 145, 203, \text{ and } 290.$$

As a consequence, we are able to prove that there are exactly 6436 (positive definite) universal quaternary forms.

The 290-Theorem is proved by recursively constructing a finite set of **escalator forms** Q' which must be contained within any universal quadratic form Q, and keeping track of the smallest number not represented by each Q' (called its **truant**). If Q' has no truant then it is universal, otherwise we extend the set of escalators by adding all (positive definite) forms containing Q' but which also represent its truant. For each escalator Q' we must either find its truant or show that it is universal.

In our case, we follow this procedure to explicitly construct a list of 6,560 quaternary escalators Q' and establish which numbers they represent via the analytic method developed in [Ha2]. This involves introducing several new computational ideas of using a **split local cover** $T \oplus dx^2 \subseteq Q'$ which allows us to reduce to checking representability using the ternary form T, and an **approximate boolean theta function** which allows us to roughly understand the numbers $\leq X$ represented by T in about time X (as opposed to the usual $X^{\frac{3}{2}}$) using equidistribution results of [D-SP]. For those Q' which are not universal, we then organize their higher-dimensional escalations in terms of escalations by a non-minimal number, which allows us to check the representation properties of only 104 additional quaternary forms. Having done this, the set of all truants gives the 29 critical numbers above. These results are also described in the Notices of the AMS [O] and Science News [P].

1.4 Total Mass of Quadratic Forms

Papers [Ha9] and [Ha10] define the **total mass** $\mathrm{TMass}_{n,\vec{\sigma}}(d)$ of rank n $\mathcal{O}_F$-valued quadratic lattices of fixed signature vector $\vec{\sigma}$ and fixed determinant squareclass d, (where $\mathcal{O}_F$ is the ring of integers of a number field F,) and investigate the behavior of $\mathrm{TMass}_{n,\vec{\sigma}}(d)$ as the determinant d varies. This is done by describing the structure of the **total mass Dirichlet series**, which in the case of positive definite

[1]The restriction to $m \in t\mathbb{Z}^2$ is due to the ineffective lower bound $L(1, \chi_t) \geq C_\varepsilon t^{-\varepsilon}$, which is intimately related to the possible existence of a Siegel zero for Dirichlet L-functions. Exceptional-type square classes are often referred to as **spinor-exceptional** square classes.

quadratic forms in n variables over $\mathbb{Z}$ is defined as

$$D_{\text{TMass},n}(s) := \sum_{d\in\mathbb{N}} \left(\sum_{\substack{\text{classes of pos. def.} \\ Q \text{ in } n \text{ variables} \\ \text{with } \det(Q)=d}} \frac{1}{|\mathrm{Aut}(Q)|} \right) d^{-s}.$$

(Here the inner sum varies over classes of positive definite $\mathbb{Z}$-valued quadratic forms in n variables with Hessian determinant $d \in \mathbb{N}$.)

In [Ha9] we establish the structural result that the total mass Dirichlet series is essentially a linear combination of two Eulerian Dirichlet series $D_{A,n}(s)$ and $D_{B,n}(s)$, each of whose Euler factors are explicitly described as a linear combination of two local mass densities with given local integral and rational invariants. When n is odd, this takes the simple form

$$D_{\text{TMass},n}(s) = \kappa_n(D_{A,n}(s) \pm D_{B,n}(s)) \tag{1}$$

for some constant κ_n, where the sign $\pm$ depends on the chosen (fixed) signature vector $\vec{\sigma}$. When n is even, the number κ_n also varies across the squareclasses $t\mathbb{N}^2$ containing d (with $t \in \mathbb{N}$ squarefree). Our main result there is stated over an arbitrary number field F for primitive quadratic lattices of fixed rank n and signature $\vec{\sigma}$, and also allows one to consider only lattices whose ambient quadratic space has specified Hasse invariants c_v at finitely many places v of F.

The main result of [Ha10] gives an explicit formula for equation (1) in the setting where $n = 3$ and F is a number field in which $p = 2$ splits completely. In the simplest case of positive definite (ternary) forms when $F = \mathbb{Q}$, we establish the explicit formula

$$D_{\text{TMass},n=3}(s) = \frac{1}{48} \cdot 2^{-s} \left[\zeta(2s-1)\zeta(s-1) - \zeta(2s-2)\zeta(s)\right]$$

for the total mass Dirichlet series by using the explicit local mass formulas in [Co-Slo1]. This gives the surprisingly simple formula

$$\mathrm{TMass}_{n=3}(d) = \frac{1}{48} \cdot \sum_{\substack{d/2=ab^2 \\ \text{with } a,b\in\mathbb{N}}} ab - b^2$$

for the total mass of positive definite $\mathbb{Z}$-valued ternary quadratic forms of Hessian determinant $d > 0$. This result can also be viewed as explicitly computing the Shintani zeta function of ternary quadratic forms, and from this perspective these formulas were obtained when $F = \mathbb{Q}$ for all n independently (and previously) by Saito and Ibukiyama [Ib-Sa] using similar methods. This makes it very reasonable that with current technology these formulas (for all n) can be extended to the context of a general number field F where $p = 2$ splits completely.

1.5 Average size of 2-torsion in class groups of n-monogenic cubic fields

In joint work with with M. Bhargava and A. Shankar [Bh-Ha-Sh], we establish exact formulas for the average size of the 2-torsion part of the class group $\mathrm{Cl}_2(K)$ of a cubic fields K, ordered by their height H, under the condition that its ring of integers $\mathcal{O}_K$ admits an index n monogenic subring $\mathbb{Z}[\alpha]$ (i.e. ring generated by one element $\alpha \in K$). This work continues a theme of proving and refining the Cohen-Lenstra heuristics for the 2-torsion in class groups of cubic fields. The Cohen-Lenstra heuristics predict the average size of the p-torsion in class groups of number fields as one varies over all fields of fixed degree, possibly with finitely many local splitting conditions (and these never change the predictions!). This paper examines the effect of the *global condition* of n-monogenicity on these heuristics, and gives the first example of family of cubic fields where *imposing local splitting conditions produces any changes in these averages*. It also provides a tool to allow precise control of these averages by carefully choosing n and splitting conditions at primes p dividing n.

More precisely, we say that such a cubic field K is **n-monogenic**, and that any two such generators α and α' are **equivalent** if $\mathbb{Z}[\alpha] = \mathbb{Z}[\alpha']$ and $\alpha' \in \alpha + \mathbb{Z}$. We define an **$n$-monogenic pair** as a pair $(K, \alpha + \mathbb{Z})$ with $[O_K : \mathbb{Z}[\alpha]] = n$ and where some $\alpha' \in \alpha + \mathbb{Z}$ has characteristic polynomial of the form $g(x) = x^3 + ax^2 + bnx + cn^2$ with $n, a, b, c \in \mathbb{Z}$. Finally, given an n-monogenic pair $(K, \alpha + \mathbb{Z})$ we define its **height** to be the maximum $H := n^2 \cdot \max\{|I(g)|^3, \frac{J(g)^2}{4}\}$ where $I(g) := n^{-\frac{2}{3}}a^2 - 3n^{\frac{1}{3}}b$

and $J(g) := -2n^{-1}a^3 + 9ab - 27nc$. In this language, we compute the average size of $\mathrm{Cl}_2(K)$ over n-monogenic pairs $(K, \alpha + \mathbb{Z})$ ordered by their height, where the field discriminant $\mathrm{Disc}(K)$ has fixed sign and where we may impose finitely many (and occasionally infinitely many compatible) splitting conditions on K. In the simplest case, we describe the average n-monogenic 2-torsion size

$$\mathbf{Cl}_{2,n,\varepsilon} := \lim_{X\to\infty} \frac{\sum_{(K,\,\alpha+\mathbb{Z})\,\in\, S_{n,\varepsilon,X}} \mathrm{Cl}_2(K)}{\# S_{n,\varepsilon,X}}$$

over the set $S_{n,\varepsilon,X}$ of n-monogenic pairs $(K, \alpha+\mathbb{Z})$ with $\gcd(n, \mathrm{Disc}(K)) = 1$, $n \neq \square$, and $\mathrm{sign}(\mathrm{Disc}(K)) = \varepsilon \in \{\pm\}$, showing that

$$\mathbf{Cl}_{2,n,\varepsilon=+} = \frac{5}{4} \qquad\qquad \mathbf{Cl}_{2,n,\varepsilon=-} = \frac{3}{2}$$

and that the same averages over the 2-torsion in the narrow class group $\mathrm{Cl}_2^+(K)$ are 2 and $\frac{3}{2}$ respectively. We also give exact formulas where we allow common factors between n and $\mathrm{Disc}(K)$, and when n is a square, in which case these averages can take many different values! Especially remarkable is the fact that *we can actually change these averages by imposing local splitting conditions* at primes p dividing n, which is a new feature not present in any other refinements of the Cohen-Lenstra averages (either with varying index n or for $n = 1$).

These results extend previous results of the proposer's collaborators [Bh-Sh1, Thrm 4.1] that count the average number of 2-torsion elements in the class groups of n-monogenic fields (ordered by height) when $n = 1$ and also as the index n is allowed to grow subject to the bound $n < X^\delta$ (for any $\delta > 0$), for heights $H < X$ as $X \to \infty$. In [Bh-Sh1] these ideas are then used to give unconditional bounds for the average rank of elliptic curves, and we expect similar applications to period-index problems for genus 1 curves to follow from these more refined n-monogenic averages.

2 Future Plans

2.1 Finiteness Theorems for Primes and Primitive Representations

Following the methods of the 290-Theorem, together with M. Bhargava, I would like to establish two further explicit finiteness theorems by producing two minimal finite sets $\mathbb{S}, \mathbb{P} \subset \mathbb{N}$ so that for any positive definite integer-valued quadratic form Q,

$$Q \text{ primitively represents } \mathbb{S} \quad\Longrightarrow\quad Q \text{ primitively represents } \mathbb{N}, \tag{2}$$

$$Q \text{ represents } \mathbb{P} \quad\Longrightarrow\quad Q \text{ represents all prime numbers.} \tag{3}$$

The first question is a natural primitive counterpart to the 290-Theorem, and seems accessible by modifying the techniques of [Bh-Ha]. The finiteness theorem for prime numbers seems more difficult, though the classically integral analogue of (3) has been proved by Bhargava [Bh2], who showed that it suffices to check all primes ≤ 73. This problem is interesting not only because it is a natural question about primes numbers, but also because certain *ternary* escalators arising in the proof may be difficult to handle. Since the analytic methods of [Ha2] are effective only for quaternary forms, this may either require the use of additional arithmetic methods as in Watson's thesis [Wa] or, failing that, may only be a conditional result based on the non-existence of certain Siegel zeros (see e.g. [Ro]).

2.2 Class numbers of Quadratic forms

The main analytic tool for investigating class numbers is Siegel's mass formula

$$\mathrm{Mass}(Q) := \sum_{Q_i \in \mathrm{Gen}(Q)} \frac{1}{\#\mathrm{Aut}(Q_i)} = \prod_v \beta_v(Q)$$

which expresses a weighted sum of the classes as a product of local densities which counts the number of ways Q represents itself over the local integers $\mathfrak{o}_v$. Since the number of automorphisms always satisfies the bounds

$$2 \leq \#\mathrm{Aut}(Q_i) \leq 2^n n!,$$

knowledge about the size of the mass of Q is equivalent to knowledge of its class number.

The paper [Ha13] uses the exact mass formula [GHY] for maximal quadratic lattices to give explicit algorithms for finding all maximal definite $\mathbb{Z}$-valued quadratic forms of bounded class number, though these techniques can also be used to enumerate all totally definite maximal $\mathcal{O}_F$-valued quadratic forms over all totally definite number fields F. The case of non-maximal lattices can be handled either in terms of maximal lattices (by understanding the size of the stabilizers of sublattices, via orthogonal buildings), or by a direct analysis of the Conway-Sloane explicit mass formula. Together these projects will provide algorithms to resolve the long-standing question of enumerating the finitely many $\mathcal{O}_F$-valued totally definite primitive quadratic lattices over totally real number fields F.

2.3 Linear relations among theta series

The question "To what extent is a positive definite quadratic form Q in n variables determined by its representation numbers $r_Q(m)$ (or equivalently, by its theta series $\Theta_Q(z)$)?" is as old as the theory of quadratic forms. In the case where $n = 2$ this follows essentially from class field theory (via Dedekind's correspondence between binary quadratic forms $Q(x, y)$ and ideal classes $\mathfrak{I}$ of imaginary quadratic number fields).

Understanding the linear span and linear independence of theta series of quadratic forms in 3 and 4 variables is a very interesting (and still open) problem. It is known from the Siegel-Weil formula that a certain weighted linear combination of theta series over all classes in a genus given by an explicit Eisenstein series, and one could hope to understand which genera of quadratic forms give the same Eisenstein series, however here the linear combinations appearing depends on the distributions of the automorphisms across the classes in these genera. One can also ask about all theta series (of fixed dimension n) satisfying a specific linear constraint, and this method was used by Schiemann [Schi] in 1997 to show the uniqueness of ternary theta series for $\mathbb{Z}$-valued quadratic forms by using a computational approach successively refining the constraints on (products of) fixed cones of reduced quadratic forms in n variables. Besides this one application, Schiemann's method has not been used to investigate any further relations. In [Co-Slo2], Conway and Sloane give an arithmetic construction of a 4-parameter family of quadratic forms in 4 variables that (for integral values of the parameter) produce the same theta series, however it is not clear how many of these quadratic forms are inequivalent.

2.4 Combinatorial Genus invariants over dyadic local fields

The theory of local integral invariants over $\mathfrak{p}$-adic fields $K_\mathfrak{p}$ for non-dyadic fields (where $\mathfrak{p} \nmid 2$) is particularly simple, and essentially states that two quadratic forms are integrally equivalent if when decomposing each as a direct sum of scaled unimodular forms, their respective unimodular components (corresponding to each possible scaling) have isomorphic quadratic spaces. This is referred to as the Jordan decomposition/equivalence theorem. For integral equivalence over $\mathbb{Q}_2$, several systems of complete invariants have been established (e.g. Jones [Jo], Pall [Pa], Conway [Co-Slo3, p378]) however the simplest of these is the "Train/Compartment" formalism of Conway that gives a combinatorial description of equivalent sets of invariants associated to a train/compartment structure. The proposer has used Conway's dyadic invariants to prove the results [Ha10] described in Section 1.3, however generalizing these results to other dyadic fields is not completely straightforward. Conway shows that his 2-adic invariants are complete by showing that they fall between the invariants of Jones and Pall, though these results only apply to the field $\mathbb{Q}_2$.

A complete set of invariants for integral equivalence of quadratic forms over dyadic fields has been worked out in a series of somewhat neglected papers of O'Meara [OM1, OM2], but his invariants are rather complicated and it is difficult to tell when two sets of invariants are equivalent. However the methods of O'Meara's papers make it clear that it is possible to formulate a theory of integral invariants in a combinatorial way similar to Conway's train/compartment description, but using invariants of the underlying quadratic spaces. This would simultaneously give a more standard system of invariants for Conway's description, and provide a generalization of integral invariants simple enough to be used to prove local enumerative results.

2.5 Period-index statistics for period 2 curves

By extending the techniques of [Bh-Sh1] and [Bh-Ha-Sh] applied to the spaces of binary quartic forms and pairs of quaternary quadratic forms, it should be possible to compute averages for the number of curves of genus 1 of period 2, with fixed index either 2 or 4.

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