Final Project: Congruence and Hasse-Weil Zeta functions

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Abstract

Many of the most important questions in number theory are connected to questions about "Zeta functions" associated to various interesting objects. In class we will see how the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ helps us to understand many deep questions about the prime numbers. It turns out that we can construct zeta functions associated to any curve (for example, the unit circle $x^2 + y^2 = 1$ or an elliptic curve $y^3 = x^3 + x$), and this helps us to understand the rational points on that curve. This Hasse-Weil zeta function is assembled by looking at the points of our curve over certain finite fields associated with each prime number (of which $\mathbb{Z}/p\mathbb{Z}$ is an example), and then multiplying these pieces together. These computations involve really interesting calculations with Gauss sums, and the resulting zeta functions are connected with current unanswered questions of number theory today. Your project should describe how to construct these zeta functions, compute a few examples simple examples, and describe their relevance to understanding the rational points on elliptic curves. **[3.5-5]**

The following is a rough outline which may be useful in thinking about/organizing your project. A very good reference is [1, Chapters 7,8,10,11,18.1–2,18.8] and also [2, Chapter 2]. It is also briefly mentioned in [3, §12.3–4]. If you have any questions about your project and/or readings, feel free to let me know, and we can setup a time to talk about it. Have Fun! =)

1. **Basics of finite fields**
2. **Definition of the Congruence zeta function**
3. **Computations for $\mathbb{A}^n$ at all primes p**
4. **Projective space**
5. **Gauss Sums and examples for some simple curves**
6. **CM Elliptic Curves and Gauss's Theorem**
7. **The Weil Conjectures**
8. **Gauss Sums and the "trivial" bound for curves in $\mathbb{A}^2$**
9. **The Hasse-Weil L-function and some conjectures**

References

[1] Kenneth Ireland and Michael Rosen. *A classical introduction to modern number theory*, volume 84 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1990.

[2] Neal Koblitz. *Introduction to elliptic curves and modular forms*, volume 97 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1993.

[3] Jeffrey Stopple. *A primer of analytic number theory*. Cambridge University Press, Cambridge, 2003. From Pythagoras to Riemann.