Final Project: Algebra and Sums of 4 squares
(Local-global principle and Class numbers)

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Abstract

In class we describe which numbers can be represented in the form x^2+y^2 or x^2+2y^2 by using certain other number systems to study them. It happens that in many cases there is no clear connection to a number system, so we must find a different approach. One idea is that we can check whether a number is represented in $\mathbb{Z}/m\mathbb{Z}$ for all numbers m, and if it is, we can conclude something about whether the number is represented. In general, we can't answer our original question without considering a few other related quadratic forms, but in some cases we can. Your project should explain how this works in general, and use it in some examples to find all numbers which are represented by certain quadratic forms. In particular, you can use this to prove that every positive integer is a sum of 4 integer squares. **[3-4]**

The following is a rough outline which may be useful in thinking about/organizing your project. A very good reference is [1, Chapters 4, 9.1, 9.5–6, 12 and Appendix A]. You can also contrast this to theorems about quadratic forms in 2 variables described in [2, §2A and §3B] and [3, §13.1]. If you have any questions about your project and/or readings, feel free to let me know, and we can setup a time to talk about it. Have Fun! =)

1. **Equivalence of quadratic forms**
2. **Local equivalence and invariants of quadratic forms**
3. **Reduction theory**
4. **Local Conditions for representability**
5. **Local-Global results and examples**

References

[1] J. W. S. Cassels. *Rational quadratic forms*, volume 13 of *London Mathematical Society Monographs*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1978.

[2] David A. Cox. *Primes of the form $x^2 + ny^2$*. A Wiley-Interscience Publication. John Wiley & Sons Inc., New York, 1989. Fermat, class field theory and complex multiplication.

[3] Jeffrey Stopple. *A primer of analytic number theory.* Cambridge University Press, Cambridge, 2003. From Pythagoras to Riemann.