In-Class Exam for Hanke – Math 2200 – 8:00am September 13, 2011

NAME (*please print*): Answer Key

Honor Code Reaffirmation: Test #1

"I have adhered to the UGA Honor Code in completing this assignment."

SIGNATURE: Jonathan Hanke

Please write your answers clearly to all problems, showing all work carefully explaining your answers. You are not allowed to use any notes, review sheets or calculators during the exam. You have exactly 75 minutes to complete the exam. Good Luck!

Problem Number	Possible Points	Points Earned:
1	30	
2	30	
3	30	
4	30	
5	30	
Total:	150	

(30) 1. Please evaluate the following limits, being sure to show all work:

(a) $\lim_{\theta\to 0} \dfrac{\theta+1}{\theta-1}$

Plug in $\theta = 0$

$$\overset{?}{=} \frac{0+1}{0-1} = \frac{1}{-1} = \boxed{-1}$$

(Ok since cts at $\theta = 0$.)

(b) $\lim_{x\to 0} \dfrac{x^2}{x^3 - x}$

Plug in $x=0$

$$\overset{?}{=} \frac{0}{0-0} = \frac{0}{0}$$ ← Indeterminate, so try something else!

Plug in $x=0$

$$= \lim_{x\to 0} \frac{x^2}{x(x^2-1)} = \lim_{x\to 0} \frac{x}{x^2-1} = \frac{0}{-1} = \boxed{0}$$

(c) $\lim_{z\to 0} \dfrac{\tan(5z)}{\cos(2z)}$

Plug in $z=0$

$$\overset{?}{=} \frac{\tan(0)}{\cos(0)} = \frac{0}{1} = \boxed{0}$$

(ok since cts at $z = 0$.)

(d) $\lim_{z\to 0} \dfrac{\tan(5z)}{\sin(2z)}$

Plug in $z=0$

$$\overset{?}{=} \frac{\tan(0)}{\sin(0)} = \frac{0}{0}$$ ← Indeterminate!

$$= \lim_{z\to 0} \frac{\sin(5z)}{\cos(5z)} \cdot \frac{1}{\sin(2z)} \cdot \frac{2z}{5z} \cdot \frac{5z}{2z} \quad \cancel{= \lim_{z\to 0}}$$

$$= \lim_{z\to 0} \underbrace{\frac{1}{\cos(5z)}}_{1} \cdot \underbrace{\frac{\sin(5z)}{5z}}_{1} \cdot \underbrace{\frac{2z}{\sin(2z)}}_{1} \cdot \underbrace{\frac{5\cancel{z}}{2\cancel{z}}}_{5/2} = \boxed{\frac{5}{2}}$$

(e) $\lim_{x\to 2} \dfrac{\sqrt{x+2}-2}{x-2}$

(Plug in $x=2$) $\Rightarrow$ $\dfrac{\sqrt{4}-2}{2-2} = \dfrac{0}{0}$.

Indeterminate!

$$= \lim_{x\to 2} \frac{\sqrt{x+2}-2}{x-2} \cdot \frac{\sqrt{x+2}+2}{\sqrt{x+2}+2} = \lim_{x\to 2} \frac{(x+2) \overset{-2}{\cancel{-4}}}{(x-2)(\sqrt{x+2}+2)} = \lim_{x\to 2} \frac{1}{\sqrt{x+2}+2}$$

(Plug in $x=2$) → $$= \frac{1}{\sqrt{4}+2} = \boxed{\frac{1}{4}}$$

(30) 2. (a) Please precisely state the (limit) definition of the derivative $f'(x)$ of a function $f(x)$ at x.

$$f'(x) := \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

(b) Please use this definition to prove that the derivative of $f(x) = \frac{1}{x^2}$ is given by $f'(x) = \frac{-2}{x^3}$.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{1}{(x+\Delta x)^2} - \frac{1}{x^2}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \cdot \left(\frac{x^2 - (x+\Delta x)^2}{x^2(x+\Delta x)^2} \right)$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \cdot \frac{\cancel{x^2} - (\cancel{x^2} + 2x \cdot \Delta x + (\Delta x)^2)}{x^2(x+\Delta x)^2}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\cancel{\Delta x}} \cdot \frac{\cancel{\Delta x}(-2x - \Delta x)}{x^2(x+\Delta x)^2}$$

$$= \lim_{\Delta x \to 0} \frac{-2x - \Delta x}{x^2(x+\Delta x^2)}$$ (Now plug in $\Delta x = 0$ to evaluate the limit!)

$$= \frac{-2x - 0}{x^2(x+0)^2} = \frac{-2x}{x^4} = \boxed{\frac{-2}{x^3}}$$

(c) Please use your results above to compute the equation of the tangent line to the graph of $f(x) = \frac{1}{x^2}$ when $x = 1$.

Point on line: $(x,y) = (1, \frac{1}{1^2}) = (1, 1)$.

Slope of line: $m = f'(1) = \frac{-2}{1^3} = -2$

Point-Slope Formula $\Longrightarrow$ $\boxed{y - 1 = -2(x-1)}$

(30) 3. (a) Please precisely define what it means for a function $f(x)$ to be continuous at $x = c$.

We say that $f(x)$ is continuous at $x = c$ if

1) $f(c)$ exists

2) $\lim_{x \to c} f(x)$ exists

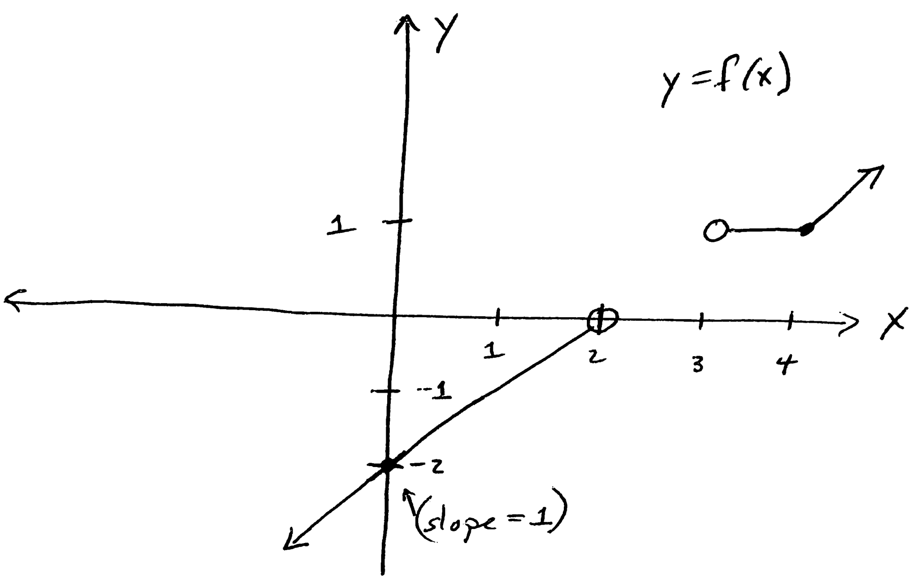
3) $\lim_{x \to c} f(x) = f(c)$.

(b) Please precisely define what it means for a function $f(x)$ to be continuous on the interval (a, b).

We say that $f(x)$ is continuous on (a, b) if $f(x)$ is continuous at every point c in (a, b).

(c) Please draw the graph of a function $f(x)$ that has the following properties:

- $f(x)$ is defined for all real numbers except when $2 \le x \le 3$.
- $f(x)$ is continuous on the intervals $(-\infty, 1)$ and $(3, \infty)$.
- $f(x)$ is *not* differentiable ~~when~~ when $x = 4$.
- $f'(0) = 1$ and $f(0) = -2$.



(30) 4. Please evaluate the following one sided limits, being sure to show all work:

(a) $\lim_{x\to 0-} \frac{|x|}{|x-2|} = \lim_{x\to 0^-} \frac{-x}{-(x-2)} = \frac{-0}{-(0-2)} = \boxed{0}$

(Plug in $x=0^-$)

$\left(\begin{array}{l} x<0 \Rightarrow |x| = -x \\ x-2<0 \Rightarrow |x-2| = -(x-2) \end{array} \right)$

(b) $\lim_{x\to 0+} \frac{|x|}{|x-2|} = \lim_{x\to 0^+} \frac{x}{-(x-2)} = \frac{0}{-(0-2)} = \boxed{0}$

(Plug in $x=0^+$)

$\left(\begin{array}{l} x>0 \Rightarrow |x| = x \\ x-2<0 \Rightarrow |x-2| = -(x-2) \end{array} \right)$

(c) $\lim_{x\to 2-} \frac{|x|}{|x-2|} = \lim_{x\to 2^-} \frac{x}{-(x-2)} = \frac{2^-}{-(2^- -2)} = \frac{2}{-(0^-)} = \frac{2}{0^+} = \boxed{\infty}$

(Plug in $x=2^-$)

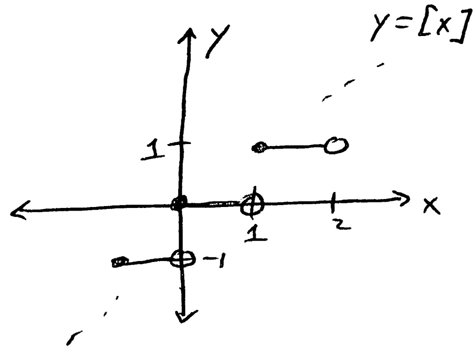
$\left(\begin{array}{l} x>0 \Rightarrow |x| = x \\ x-2<0 \Rightarrow |x-2| = -(x-2) \end{array} \right)$

(d) $\lim_{x\to 2+} \frac{|x|}{|x-2|} = \lim_{x\to 2^+} \frac{x}{x-2} = \frac{2^+}{2^+ -2} = \frac{2}{0^+} = \boxed{\infty}$

(Plug in $x=2^+$)

$\left(\begin{array}{l} x>0 \Rightarrow |x| = x \\ x-2>0 \Rightarrow |x-2| = x-2 \end{array} \right)$

In the following two problems, the function $[x]$ is the "greatest integer" function we discussed in class, that is defined to be the largest integer $\leq x$.



(e) $\lim_{x\to 0-} \frac{[x]}{x} = \frac{-1}{0^-} = -(-\infty) = \boxed{\infty}$

(Plug in 0^-)

(f) $\lim_{x\to 0+} \frac{[x]}{x} = \frac{0}{0^+} = \boxed{0}$

(Plug in 0^+)

 Name: ______________________________

(30) 5.

(a) Please precisely state the Intermediate Value Theorem.

Suppose that $f(x)$ is a continuous function on the closed interval $[a,b]$. Then for every k between $f(a)$ and $f(b)$, there is some c in $[a,b]$ so that $f(c)=k$.

(b) Please use the Intermediate Value Theorem to show that the function $f(x) = 2\sin(x) - x + 1$ has a root in the interval $[0, \pi]$.

Since $\sin(x)$ and $-x+1$ are both continuous functions at all real numbers, we see that $f(x)$ is continuous on $[0,\pi]$.

Also $f(0) = 2\sin(0) - 0 + 1 = 1 > 0$

and $f(\pi) = 2\sin(\pi) - \pi + 1 = 1 - \pi < 0$,

so taking $k=0$, $\underbrace{\quad}_{a=0 \text{ and } b=\pi,}$ we see that k is between $f(a)$ and $f(b)$.

Therefore by the Intermediate Value Theorem we know that $f(c)=0$ for some c in $[0,\pi]$, and c is a root of $f(x)$.