Final Project: Cubic and Biquadratic reciprocity

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Abstract

In class we will be interested in exploring the question "When is α a square in $\mathbb{Z}_p$?" The answer is intimately connected to the famous theorem of *quadratic reciprocity*, originally proved by Gauss (in many different ways). Your project should explain how one can answer the question of "When is α a cube (or 4^{th} power) in $\mathbb{Z}_p$?", and prove the associated 'higher' reciprocity law(s) of cubic (and biquadratic) reciprocity. **[4-5]**

The following is a rough outline which may be useful in thinking about/organizing your project. Good references are [5, Chapters 5, 6, 7.3, 9], [1, §1C–D, §4], [4, Chapter 11], [2], [3, ??]. If you have any questions about your project and/or readings, feel free to let me know, and we can setup a time to talk about it. Have Fun! =)

1. **The Legendre symbol**
2. **The statement of quadratic reciprocity**
3. **Proof of Quadratic reciprocity by Gauss Sums**
4. **Applications of Quadratic reciprocity**
5. **The cubic residue symbol**
6. **The ring** $\mathbb{Z}[\omega]$
7. **The statement of cubic reciprocity**
8. **Proof of Cubic Reciprocty by Gauss sums**
9. **Applications and connection with** $p = x^2 + 27y^2$
10. **Comments about Biquadratic Reciprocity and** $\mathbb{Z}[i]$

References

[1] David A. Cox. *Primes of the form $x^2 + ny^2$*. A Wiley-Interscience Publication. John Wiley & Sons Inc., New York, 1989. Fermat, class field theory and complex multiplication.

[2] H. Davenport. *The higher arithmetic*. Cambridge University Press, Cambridge, seventh edition, 1999. An introduction to the theory of numbers, Chapter VIII by J. H. Davenport.

[3] Harold M. Edwards. *Fermat's last theorem*, volume 50 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1996. A genetic introduction to algebraic number theory, Corrected reprint of the 1977 original.

[4] Jay R. Goldman. *The queen of mathematics*. A K Peters Ltd., Wellesley, MA, 1998. A historically motivated guide to number theory.

[5] Kenneth Ireland and Michael Rosen. *A classical introduction to modern number theory*, volume 84 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1990.