

Final Project: Arithmetic and sums of 4 squares (Quaternion algebras)

Written by Jonathan Hanke

February 21, 2007

Abstract

In class we will see that the question of which numbers have the form $x^2 + y^2$ has deep connections to certain other number systems. A similar connection exists for the question of which numbers can be written a sum of 4 squares, and it related to a non-commutative number system known as a *quaternion algebra*. The simplest example of this is the quaternions, discovered by Hamilton in the 1800's, is numbers of the form $a + bi + cj + dk$ where $a, b, c, d \in \mathbb{Z}$, and $i^2 = j^2 = k^2 = -1$. Your project should describe this connection, and use it to answer questions related to which numbers can be written as sums of 4 squares.

The following is a rough outline which may be useful in thinking about/organizing your project. The only reasonable reference I could find was [1, pp303–310], but it has a very complete discussion. If you have any questions about your project and/or readings, feel free to let me know, and we can setup a time to talk about it. Have Fun! =)

1. **2 kinds of integral quaternions**
2. **Quaternion norms and multiplicativity**
3. **Definitions of divisibility, units, and associates**
4. **Right division, right ideals, and right GCDs**
5. **Prime quaternions and primes in \mathbb{Z}**
6. **Moving between quaternions with associates**
7. **Other quadratic forms and quaternion algebras with UPF**

References

- [1] G. H. Hardy and E. M. Wright. *An introduction to the theory of numbers*. The Clarendon Press Oxford University Press, New York, fifth edition, 1979.

Copyright Jonathan Hanke © 2007, 2011

<http://www.jonhanke.com>