Final Project: Pell's equation and the continued fraction of $\sqrt{D}$

Written by Jonathan Hanke

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Abstract

One of the simplest and oldest quadratic equations in number theory are the (slightly misnamed) Pell equations $x^2 - Dy^2 = \pm 1$ for some $D \in \mathbb{Z} > 0$. These solutions (x, y) have close connections to the quadratic number field $\mathbb{Q}(\sqrt{D})$, but are much harder to understand than their positive counterparts $x^2 + Dy^2 = 1$. Some information about them can be found be studying the continued fraction expansion of $\sqrt{D}$, though there is still much left to know. Your project should describe this connection, and use it to understand the solutions of these equations, and what they can tell us about their associated quadratic number fields. **[3-4]**

The following is a rough outline which may be useful in thinking about/organizing your project. Good general references are [1, Chapter IV], [2, §1.9 and §8.2], and [4]. There is also a brief but interesting commentary in [3, pp192–193 and §17.5].

If you have any questions about your project and/or readings, feel free to let me know, and we can setup a time to talk about it. Have Fun! =)

1. **The cyclic method of solving Pell's equation** $x^2 - Dy^2 = 1$

   There is an interesting historical discussion in [2, pp25–36] which describes the cyclic method and has some interesting tables.

2. **The continued fraction expansion of** $sqrtD$ **and properties of the convergents.**

3. **Relationship to the units in** $\mathbb{Q}(\sqrt{D})$

4. **Solvability of** $x^2 - Dy^2 = -1$ **and** $x^2 - Dy^2 = -4$

5. **Solving other 'Pell-type' equations**

References

[1] H. Davenport. *The higher arithmetic*. Cambridge University Press, Cambridge, seventh edition, 1999. An introduction to the theory of numbers, Chapter VIII by J. H. Davenport.

[2] Harold M. Edwards. *Fermat's last theorem*, volume 50 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1996. A genetic introduction to algebraic number theory, Corrected reprint of the 1977 original.

[3] Kenneth Ireland and Michael Rosen. *A classical introduction to modern number theory*, volume 84 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1990.

[4] Richard A. Mollin. *Quadratics*. CRC Press Series on Discrete Mathematics and its Applications. CRC Press, Boca Raton, FL, 1996.