PROMYS 2009 Research Problems

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One classical problem that we spend a lot of time thinking about at PROMYS is "What integers can be written in the form $Q_d(x, y) := x^2 + dy^2$ for some fixed integer d?". This question has several natural generalizations, and each of the research problems here will try to ask similar questions for other related kinds of forms for writing integers.

1 Explicit formulas for sums of more squares

One way of thinking about $Q_d(x, y)$ is as a **quadratic form**, which we define to be a polynomial all of whose terms have degree two (i.e. there are exactly two letters appearing in each term, counting multiplicity). In general, an integer-valued quadratic form can have n variables and can be written as

$$Q(\vec{x}) := Q(x_1, \ldots, x_n) := \sum_{1 \leq i \leq j \leq n} c_{ij} x_i x_j$$

with integer coefficients c_{ij}. A natural related (but more complicated) question is to fix a quadratic form $Q(\vec{x})$ and ask "What integers can written as (or represented by) $Q(\vec{x})$?". This question has a particularly nice answer for certain kinds of quadratic forms described as having 'class number one'. These forms are so nice that we can easily decide not only which numbers are represented by $Q(\vec{x})$, but also we can obtain *exact formulas* for the number of representations, which we denote by $r_Q(m)$. More formally, we define

$$r_Q(m) := \#\{\vec{x} \in \mathbb{Z}^n \mid Q(\vec{x}) = m\}$$

and notice that this is only finite for $n \geq 2$ when Q takes on only values ≥ 0 and the value zero only when $\vec{x} = \vec{0}$. (This condition for $n \geq 2$ is usually referred to by saying Q is **positive definite**.) An example of this is the sum of four squares form $Q(\vec{x}) = x_1^2 + x_2^2 + x_3^2 + x_4^2$, for which we know the exact formula

$$r_Q(m) = 8 \cdot \sum_{0 < d \mid m, 4 \nmid d} d$$

when $m > 0$.

One way of proving this formula is to try to relate Q to the norm form of some 4-dimensional object, in a similar manner to the way we understand the numbers represented by x^2+y^2 by viewing it as the norm form from $\mathbb{Z}[i]$ (and similarly for $\mathbb{Q}_d(x,y)$ as the norm form from $\mathbb{Z}[\sqrt{-d}]$). However in our case the 4-dimensional object (called a **quaternion ring**) happens to not be commutative! The goal of this project is to understand the proof of the exact formula for sums of 4 squares, and then to generalize this to understand the numbers represented by one of the other class number one forms.

References:

1. Hardy and Wright – "Introduction to the Theory of Numbers"
2. Conway and Smith book – "Quaternions and Octonions"
3. Princeton Ph.D. Thesis of Donkar
4. Brzezinski – Definite quaternion orders of class number one (1995)

2 Finiteness Theorems for Quadratic and Hermitian forms

Another way of generalizing the question of what numbers are represented by $Q_d(x,y)$ is to explicitly think of them as related to norm forms $N(\alpha) := \alpha\bar{\alpha}$ from the integers in the quadratic field $\mathbb{Q}(\sqrt{-d})$. This way of thinking leads one to consider Q_d as a special case of **hermitian forms**, which are quadratic forms with a built-in conjugation for one of the two variables in each degree two term. In general, a hermitian form in n variables can be written as

$$H(\vec{\alpha}) := \sum_{1\leq i,j\leq n} \gamma_{ij}\alpha_i\bar{\alpha}_j$$

Because a hermitian form always has a conjugation, it is defined relative to a given quadratic number field $K := \mathbb{Q}(\sqrt{-d})$ giving the conjugation, and so we take the α_i to be in the ring of integers $\mathcal{O}_\mathcal{K}$ of K and require that $H(\vec{\alpha})$ take on only integer values. Notice that the hermitian form $H(\alpha) := \alpha\bar{\alpha}$ is just the familiar quadratic form $Q_d(x,y)$ if we write $\alpha = x + y\sqrt{-d}$. We can then ask what integers are represented by any given hermitian form, in the same way we did for quadratic forms. (Say as a sum of 2 norms!)

In addition to trying to understand what numbers are represented by a given (positive definite) quadratic/hermitian form, we can try to understand what quadratic/hermitian forms represent all natural numbers. (It is not hard to check that we must have $d > 0$ in order for H to have a chance of being positive definite.) One way of answering this question is in terms of a **finiteness theorem**, which says that if a form represents a finite set of numbers, then it must represent all natural numbers. For integer-valued quadratic forms, it is enough to check that a positive definite quadratic form $Q(\vec{x})$ represents all numbers up to 290 (actually just 29 of them) to ensure that it represents all natural numbers, but it is not known what the finite set of numbers we need to check to ensure a (positive definite) hermitian form represents all natural numbers. We can look for a finiteness theorem either for all hermitian forms for a given $d > 0$, or for all hermitian forms for all $d > 0$ at once. The goal of this project is to use known results about quadratic forms to conjecture and prove explicit finiteness theorems for hermitian forms.

References:

1. Conway and Bhargava papers on finiteness theorems for quadratic forms
2. Books on Quadratic and Hermitian forms – Scharlau, Knus, Shimura's books
3. Kim and Park – A few uncaught universal hermitian forms (2007)
4. Iwabuchi – Universal Binary Positive Definite Hermitian lattices (2000)
5. Kominers – On Universal Binary Hermitian Forms (2009)